A NEW GENERALIZATION OF THE TAKAGI FUNCTION

KAZUKI OKAMURA

ABSTRACT. We consider a one-parameter family of functions $\{F(t,x)\}_t$ on $[0,1]$ and partial derivatives $\partial_t^k F(t,x)$ with respect to the parameter t. Each function of the class is defined by a certain pair of two square matrices of order two. The class includes the Lebesgue singular functions and other singular functions. Our approach to the Takagi function is similar to Hata and Yamaguti. The class of partial derivatives $\partial_t^k F(t,x)$ includes the original Takagi function and some generalizations. We consider real-analytic properties of $\partial_t^k F(t,x)$ as a function of x, specifically, differentiability, the Hausdorff dimension of the graph, the asymptotic around dyadic rationals, variation, a question of local monotonicity and a modulus of continuity. Our results are extensions of some results for the original Takagi function and some generalizations.

1. INTRODUCTION

The Takagi function [14], which is denoted by T throughout the paper, is an example of continuous nowhere differentiable functions and has been considered from various points of view. Since T is a fractal function, it is interesting to investigate real-analytic properties of T. For example, differentiability, the Hausdorff dimension of the graph, the asymptotic around dyadic rationals and a modulus of continuity of T have been considered.

Hata and Yamaguti [6] showed the following relation between the Takagi function $T(x)$ and the Lebesgue singular[1] function $L_a(x)$ with singularity parameter a :

$$\left.\frac{\partial}{\partial a}\right|_{a=1/2} L_a(x) = T(x). \tag{1.1}$$

Now give a precise definition of L_a. Let μ_a be the probability measure on $\{0,1\}$ with $\mu_a(\{0\}) = a$ and $\mu_a^{\otimes\mathbb{N}}$ be the product measure of μ_a on $\{0,1\}^{\mathbb{N}}$. Let $\varphi : \{0,1\}^{\mathbb{N}} \to [0,1]$ be a function defined by $\varphi((x_n)_n) = \sum_{n=1}^{\infty} x_n/2^n$. Let L_a be the distribution function of the image measure of $\mu_a^{\otimes\mathbb{N}}$ by φ. L_a is identical with $\Phi_{2,1/a}$ in Paradis, Viader and Bibiloni [11].

Recently, de Amo, Díaz Carrillo and Fernández-Sánchez [3] considered $\partial_a^n L_a(x)$ at $a \neq 1/2$. (Here and henceforth ∂_z^n denotes the n-th partial derivatives with respect to the variable z. If $n = 1$ write simply ∂_z.) They showed for any $a \neq 1/2$ and for $n \geq 1$, $\partial_a^n L_a(x)$ has zero derivative at almost every x. They claimed if n is odd, $\partial_a^n L_a$ is of monotonic type on no open interval (MTNI[2]). That is, on any open interval J in $[0,1]$,

$$-\infty = \inf_{x,y\in J, x\neq y} \frac{\partial_a^n L_a(x) - \partial_a^n L_a(y)}{x-y} < \sup_{x,y\in J, x\neq y} \frac{\partial_a^n L_a(x) - \partial_a^n L_a(y)}{x-y} = +\infty.$$

In this paper we consider a further generalization of T by replacing L_a in (1.1) with more general functions and parametrizations. The author's paper [10] considers a probability measure μ_{A_0,A_1} on $[0,1]$ defined by a certain pair of two 2×2 real matrices (A_0, A_1). μ_{A_0,A_1} is singular or absolutely continuous with respect to the Lebesgue measure. The class of probability measures in [10] contains not only the Bernoulli measures but also many *non-product* measures[3]. Parametrize (A_0, A_1) by a parameter t around 0. Assume each component of $A_0(t)$

2000 *Mathematics Subject Classification.* Primary : 26A27; secondary : 39B22; 60G42; 60G30.

[1]In this paper a singular function is a continuous increasing function on $[0,1]$ whose derivatives are zero Lebesgue-a.e.

[2]We follow Brown, Darji and Larsen [5] for this terminology.

[3]We identify $[0,1)$ with the Cantor space $\{0,1\}^{\mathbb{N}}$ in the natural way. We consider non-atomic measures on $[0,1]$ only and we do not need to distinguish $[0,1]$ from $[0,1)$.

and $A_1(t)$ is smooth[4] with respect to t and $(A_0(0), A_1(0)) = (A_0, A_1)$. Denote the distribution function of μ_t by $F(t, \cdot)$. That is, $F(t,x) = \mu_{A_0(t),A_1(t)}([0,x]), x \in [0,1]$.

The main subject of this paper is investigating real analytic properties for the k-th partial derivative $f_k(x) := \partial_t^k F(0,x)$. Our framework gives a generalization of T. $F(t,x) = L_{a+t}(x)$ for a specific choice of $(A_0(t), A_1(t))$. Thus our framework contains the one of [3]. Our generalization is different from the ones by [6] and Kôno [7]. The graphs of these curves can be quite different, from Takagi's classical, T, to very *asymmetrical* ones as shown in figures 1 and 2 below. In Section 2 we will give the framework and show f_k is well-defined and continuous on $[0,1]$ for each $k \geq 1$.

In Section 3 we will show the Hausdorff dimension of the graph of f_k is 1. This extends Allaart and Kawamura [1, Corollary 4.2] and is applicable to the framework in [3, Section 5]. Our proof is different from Mauldin and Williams [9] and [1] and seems simpler than them because we do not need to investigate strength of continuity of f_k. In Section 4 we will show the derivative of f_k is 0 almost everywhere. This extends [3, Theorems 12 and 13]. We will examine the asymptotic of f_k around dyadic rationals in Section 5. The asymptotic of f_k around *dyadic rationals* and around *Lebesgue-a.e. points* can be similar on the one hand but can be considerably different on the other hand. As shown in Figure 1 there is a fractal function whose derivatives are zero at all dyadic rationals. To our knowledge such a fractal function is unusual.

If we consider the case $k = 1$ and the "linear" case, each of which contains the original Takagi function T, we have more sophisticated results. In Theorem 6.2 we will consider differentiability and variation of f_k. [3, Theorem 14] states if we consider the "linear" case and k is odd, f_k is MTNI. Theorem 6.3 will extend [3, Theorem 14] to *all* $k \geq 1$. If μ_0 is singular, the asymptotic of f_k around *μ_0-a.s. points* and around *Lebesgue-a.e. points* can be considerably different. In Section 7 we will consider a modulus of continuity of f_1. Theorem 7.3 will extend Allaart and Kawamura [2, Theorem 5.4], which gives a necessary and sufficient condition for the existence of

$$\lim_{h \to 0} \frac{T(x+h) - T(x)}{h \log_2(1/|h|)} \quad \text{at non-dyadic } x.$$

Theorem 7.7 will investigate a modulus of continuity of f_1 at *μ_0-a.s. points*. It is similar to [7]. We have the original Takagi function case[5] of [7] by our approach. Our proofs are different from [2] and [7]. We do not use [7, Lemma 3] which plays an important role in [2] and [7].

2. Framework

2.1. Definition of μ_{A_0,A_1}. Let $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $i = 0,1$, be two real 2×2 matrices such that the following hold :

(i) $0 = b_0 < \dfrac{a_0 + b_0}{c_0 + d_0} = \dfrac{b_1}{d_1} < \dfrac{a_1 + b_1}{c_1 + d_1} = 1$.

(ii) $a_i d_i - b_i d_i > 0$, $i = 0,1$.

(iii) $(a_i d_i - b_i d_i)^{1/2} < \min\{c_i, c_i + d_i\}$, $i = 0,1$.

Consider a functional equation for $f : [0,1] \to \mathbb{R}$:

$$f(x) = \begin{cases} \Phi\left(A_0, f(2x)\right) & 0 \leq x \leq 1/2 \\ \Phi\left(A_1, f(2x-1)\right) & 1/2 \leq x \leq 1, \end{cases} \quad \text{where } \Phi(A,z) := \frac{az+b}{cz+d} \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (2.1)$$

Conditions (i) - (iii) assure the existence of a unique continuous solution for (2.1). (2.1) is a special case of de Rham's functional equations [12]. Let μ_{A_0,A_1} be the measure such that the unique continuous solution f of (2.1) is the distribution function of μ_{A_0,A_1}. By conditions (i) - (iii) we can represent all components of A_0, A_1 by b_1, c_0 and c_1. We can assume $d_0 = d_1 = 1$.

[4] In this paper a smooth function is a function differentiable infinitely many times.

[5] [7] considers this in a general setting different from ours.

Conditions (i) - (iii) imply $a_0 = b_1(c_0+1)$, $b_0 = 0$, $a_1 = 1 - b_1 + c_1$ and

$$b_1 \in (0,1), c_0 \in \left(b_1 - 1, \frac{1}{b_1} - 1\right), c_1 \in \left(-b_1, \frac{b_1}{1-b_1}\right). \tag{2.2}$$

If $b_1 = a$ and $c_0 = c_1 = 0$ the Lebesgue singular function L_a is the distribution function of μ_{A_0,A_1}. $c_0 = c_1 = 0$ if and only if both $\Phi(A_0;\cdot)$ and $\Phi(A_1;\cdot)$ are linear functions. By [10, Theorem 1.2], μ_{A_0,A_1} is absolutely continuous if $c_0 = (2b_1)^{-1} - 1$ and $c_1 = 1 - 2b_1$, and singular otherwise. Let

$$\alpha := \min\left\{0, \frac{c_0}{1-b_1(c_0+1)}, \frac{c_1}{b_1}\right\} \text{ and } \beta := \max\left\{0, \frac{c_0}{1-b_1(c_0+1)}, \frac{c_1}{b_1}\right\}.$$

$\alpha = \beta = 0$ if and only if $c_0 = c_1 = 0$. Roughly speaking α and β measure how μ_{A_0,A_1} is "far" from the Bernoulli measures.

Now define a "dual" $(\widetilde{A}_0, \widetilde{A}_1)$ associated with (A_0, A_1) in order to shorten some proofs.

Definition 2.1 (Dual matrices)**.** Let

$$(\widetilde{b}_1, \widetilde{c}_0, \widetilde{c}_1) := \left(1 - b_1, -\frac{c_1}{1+c_1}, -\frac{c_0}{1+c_0}\right), \tag{2.3}$$

Define $\widetilde{A}_i, i = 0,1$, $\widetilde{\alpha}$ and $\widetilde{\beta}$ by substituting $(\widetilde{b}_1, \widetilde{c}_0, \widetilde{c}_1)$ for (b_1, c_0, c_1) in the definition of A_i, α and β. (2.2) holds for $(\widetilde{A}_0, \widetilde{A}_1)$ if and only if it holds for (A_0, A_1). We have

$$\mu_{\widetilde{A}_0,\widetilde{A}_1}([0,x]) = \mu_{A_0,A_1}([1-x,1]),\ x \in [0,1]. \tag{2.4}$$

$$\widetilde{\widetilde{A}}_i = A_i,\ i = 0,1. \tag{2.5}$$

2.2. **Parametrization.** (1) In addition to (2.2) we assume either the Lipschitz constant of $\Phi\left({}^tA_1; y\right)$ on $y \in [\alpha, \beta]$ or the Lipschitz constant of $\Phi\left({}^t\widetilde{A}_1; y\right)$ on $y \in \left[\widetilde{\alpha}, \widetilde{\beta}\right]$ is strictly less than 1. That is

$$(1+c_1)\left(1 - b_1(1+c_0)\right)^2 < 1 - b_1 \text{ or } (b_1+c_1)^2 < b_1(1+c_0)(1+c_1). \tag{2.6}$$

Assume this condition by a difficulty arising in computation in Lemma 2.5 below. However if $c_0 = c_1 = 0$, (2.6) holds. The Lipschitz constant of $\Phi\left({}^tA_0; y\right)$ on $y \in [\alpha, \beta]$ and the Lipschitz constant of $\Phi\left({}^t\widetilde{A}_0; y\right)$ on $y \in \left[\widetilde{\alpha}, \widetilde{\beta}\right]$ are strictly less than 1.

(2) Conditions (2.2) and (2.6) define an open set E in $\mathbb{R}^3$ in which we will consider different curves.

$$E := \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, 0 < x < 1, x - 1 < y < \frac{1-x}{x}, -x < z < \frac{x}{1-x}\right\}$$
$$\cap \left\{(x,y,z) \middle| (1+z)(1-x(1+y))^2 < 1 - x \text{ or } (x+z)^2 < x(1+y)(1+z)\right\}.$$

(3) Fix a point $(b_0, c_0, c_1) \in E$. We consider a smooth curve $(b_1(t), c_0(t), c_1(t))$ in E on an open interval containing 0 such that $(b_1(0), c_0(0), c_1(0)) = (b_0, c_0, c_1)$.

Define $A_0(t), A_1(t), \alpha(t), \beta(t)$ by substituting $(b_1(t), c_0(t), c_1(t))$ for (b_1, c_0, c_1) in the definition of A_0, A_1, α, β. Let

$$\mu_t := \mu_{A_0(t),A_1(t)} \text{ and } F(t,x) := \mu_t([0,x]), \quad x \in [0,1].$$

This class of smooth curves includes the frameworks of [1], [2], [3] and [6]. We have

$$\{(x,0,0) : 0 < x < 1\} \subset E.$$

If $(b_1(t), c_0(t), c_1(t)) = (t + a, 0, 0)$, $F(0,x) = L_a(x)$.

2.3. **Notation and lemma.** Let $X_i(x) := z_i$ if $x = \sum_{n \ge 1} 2^{-n} z_n$ is the dyadic expansion of x[6].

Definition 2.2. (i)

$$G_j(t,y) := \Phi\left({}^t A_j(t); y\right), \ y \in [\alpha(t), \beta(t)], j = 0, 1. \tag{2.7}$$

(ii)

$$p_0(t,y) := \frac{y+1}{y + b_1(t)^{-1}} \text{ and } p_1(t,y) := 1 - p_0(t,y), \ y \in [\alpha(t), \beta(t)].$$

(iii) Let $p_{min}(t)$ and $p_{\max}(t)$ be the minimum and maximum of $\{p_0(t, \alpha(t)), p_1(t, \beta(t))\}$.
(iv)

$$g_0(t,x) := 0 \text{ and } g_i(t,x) := G_{X_i(x)}\left(t, g_{i-1}(t,x)\right), \quad x \in [0,1), i \ge 1.$$

(v)

$$H_n(t,x) := p_{X_{n+1}(x)}\left(t, g_i(t,x)\right), \quad P_n(t,x) := p_0\left(t, g_n(t,x)\right), \quad x \in [0,1).$$

(vi)

$$M_n(t,x) := \prod_{i=0}^{n-1} H_i(t,x), \quad x \in [0,1).$$

Example 2.3. If $c_0(t) = c_1(t) = 0$ then for $x \in [0,1)$

$$G_0(t,x) = b_1(t)x, \quad G_1(t,x) = (1 - b_1(t))x.$$

$$\alpha(t) = g_n(t,x) = \beta(t) = 0, \quad n \ge 0.$$

$$p_0(t,0) = P_n(t,x) = b_1(t) = 1 - p_1(t,0), \quad n \ge 0.$$

$$p_{min}(t) = \min\{b_1(t), 1 - b_1(t)\} \text{ and } p_{max}(t) = \max\{b_1(t), 1 - b_1(t)\}.$$

$$H_n(t,x) = b_1(t) 1_{\{X_{n+1}(x)=0\}}(x) + (1 - b_1(t)) 1_{\{X_{n+1}(x)=1\}}(x).$$

$$M_n(t,x) = b_1(t)^{a_{n,0}} (1 - b_1(t))^{n - a_{n,0}} \text{ where } a_{n,0} := \left|\{1 \le i \le n : X_i(x) = 0\}\right|.$$

In this case we do not need to introduce G, g, p, P, H and M. However we would like to consider the case that $c_0(t) = c_1(t) = 0$ *fails*. G_i, g_n, p_i, H_n and M_n are defined in order to give a useful expression for $F(t,x)$ in (2.11) below.

The following are easy to see so the details are left to readers.

Lemma 2.4. *For $n \ge 0$ and $x \in [0,1)$*
(i)

$$\alpha(t) \le g_n(t,x) \le \beta(t), \tag{2.8}$$

(ii)

$$0 < p_{min}(t) \le H_n(t,x) \le p_{max}(t) < 1. \tag{2.9}$$

(iii)

$$\mu_t\left([x_n, x_n + 2^{-n})\right) = M_n(t,x). \tag{2.10}$$

(iv)

$$F(t,x) = \sum_{n=0}^{+\infty} X_{n+1}(x) \left(M_n(t,x) - M_{n+1}(t,x)\right). \tag{2.11}$$

By (i) $g_n(t,x), H_n(t,x), P_n(t,x)$ and $M_n(t,x)$ are well-defined for any n and x.

Define $(\widetilde{b}_1(t), \widetilde{c}_0(t), \widetilde{c}_1(t))$ and $(\widetilde{A}_0(t), \widetilde{A}_1(t))$ by substituting $(b_1(t), c_0(t), c_1(t))$ in Definition 2.1. By (2.5) $(\widetilde{b}_1(t), \widetilde{c}_0(t), \widetilde{c}_1(t))$ is also a smooth curve in E. Define $\widetilde{\mu}_t, \widetilde{F}, \widetilde{G}_j, \widetilde{g}_n, \widetilde{p}_j, \widetilde{P}_i, \widetilde{H}_i, \widetilde{M}_n, \widetilde{p}_{\min}$ and $\widetilde{p}_{\max}$ in the same manner by substituting $(\widetilde{b}_0, \widetilde{c}_0, \widetilde{c}_1)$ for (b_0, c_0, c_1). Lemma 2.4 hold also for $\widetilde{g}_i, \widetilde{H}_n, \widetilde{M}_n, \widetilde{\mu}_t, \widetilde{p}_{\min}$ and $\widetilde{p}_{\max}$.

[6]As usual we assume the number of n with $z_n = 1$ is finite.

2.4. **Well-definedness and continuity of f_k.** (2.6) has been introduced in order to establish a uniform boundedness for $\partial_t^k H_n(t,x)$ as follows.

Lemma 2.5. *For any $k \ge 0$ there is a continuous function $C_{1,k}(t)$ on a neighborhood of $t=0$ such that for each t in the neighborhood :*

$$\sup_{n\ge 0, x\in(0,1)} \left|\partial_t^k H_n(t,x)\right| \le C_{1,k}(t)$$

Proof. The case $k=0$ follows from (2.9). Assume $k\ge 1$. Then $\left|\partial_t^k H_n(t,x)\right| = \left|\partial_t^k P_n(t,x)\right|$.

Recall (2.6). Assume $(1+c_1)(1-b_1(1+c_0)) < 1-b_1$. Then

$$(1+c_1(t))(1-b_1(t)(1+c_0(t))) < 1-b_1(t) \tag{2.12}$$

holds if t is close to 0.

Since $\partial_t^k P_n(t,x)$ is a multivariate polynomial consisting of

$$\partial_t^j g_n(t,x) \text{ and } \partial_t^{j'}\partial_y^{j''} P_n(t,x),\ 0\le j,j',j''\le i$$

as variables, it suffices to show that for each $k\ge 1$ there is a continuous function $C_{2,k}(t)$ such that

$$\sup_{n\ge 0, x\in(0,1)} \left|\partial_t^k g_n(t,x)\right| \le C_{2,k}(t) < +\infty. \tag{2.13}$$

We now show (2.13) by induction on k. The case $k=0$ follows from (2.8). Assume (2.13) holds for $k=0,1,\dots i-1$. Then

$$\partial_t^i g_n(t,x) = \partial_y G_{X_n(x)}(t, g_{n-1}(t,x))\partial_t^i g_{n-1}(t,x) + \mathrm{Poly}(i,n).$$

Here $\mathrm{Poly}(i,n)$ is a multivariate polynomial consisting of

$$\partial_t^j g_{n-1}(t,x) \text{ and } \partial_t^{j'}\partial_y^{j''} G_{X_n(x)}(t, g_{n-1}(t,x)),\ 0\le j,j',j''\le i-1$$

as variables.

By the hypothesis of induction and (2.8), for each i, there is a continuous function $C_{3,i}(t)$ such that

$$\left|\partial_t^i g_n(t,x)\right| \le \left(\max_{l\in\{0,1\}, y\in[\alpha(t),\beta(t)]} \partial_y G_l(t,y)\right)\left|\partial_t^i g_{n-1}(t,x)\right| + C_{3,i}(t).$$

By (2.12)

$$\max_{l\in\{0,1\}, y\in[\alpha(t),\beta(t)]} \partial_y G_l(t,y) < 1.$$

Therefore

$$\sup_{n\ge 0, x\in(0,1)} \left|\partial_t^i g_n(t,x)\right| \le \frac{C_{3,i}(t)}{1-\max_{l\in\{0,1\}, y\in[\alpha(t),\beta(t)]} \partial_y G_l(t,y)}.$$

Hence (2.13) holds.

Second assume $(b_1+c_1)^2 < b_1(1+c_0)(1+c_1)$. By (2.3) and continuity of $(b_1(t), c_0(t), c_1(t))$

$$(1+\widetilde{c}_1(t))\left(1-\widetilde{b}_1(t)(1+\widetilde{c}_0(t))\right) < 1-\widetilde{b}_1(t)$$

holds if t is close to 0. The rest of the proof goes in the same manner as above. □

Let

$$x_n := \sum_{i=1}^{n} \frac{X_i(x)}{2^i},\quad x\in[0,1) \quad\text{and}\quad D := \bigcup_{n\ge 1}\left\{\frac{k}{2^n}\ \middle|\ 1\le k\le 2^n-1\right\}.$$

Theorem 2.6. (i) *For any $k\ge 0$ there is $C_k>0$ such that*

$$\left|\frac{\partial_t^k F(0,x_n+2^{-n}) - \partial_t^k F(0,x_n)}{F(0,x_n+2^{-n}) - F(0,x_n)}\right| \le C_k n^k,\ x\in[0,1), n\ge 1. \tag{2.14}$$

(ii) $\partial_t^k F(0,x)$ *is well-defined for any* $x \in [0,1] \setminus D$.
(iii) *Let* C_k *be the constant above. Then*

$$\left| \frac{\partial_t^k F(0,x) - \partial_t^k F(0,y)}{F(0,x) - F(0,y)} \right| \le C_k(-\log_2 |x-y|)^k, \ x \neq y. \tag{2.15}$$

Now we can define

$$f_k(x) := \partial_t^k F(0,x) \text{ and } \Delta_k F(x,y) := \frac{\partial_t^k F(0,x) - \partial_t^k F(0,y)}{F(0,x) - F(0,y)}, \ x \neq y, \ k \ge 0.$$

By (2.15), f_k is continuous and if μ_0 is absolutely continuous

$$|f_k(x) - f_k(y)| = O\left(|x-y| \left(-\log_2 |x-y|\right)^k\right). \tag{2.16}$$

Whether (2.15) is best or not will be discussed after Theorem 5.4. The key of the proof of (i) is giving an upper bound for $\left|\partial_t^l H_j(t,x)\right|$ uniform with respect to x by Lemma 2.5. For (ii), roughly speaking, the key is showing the exchangeability of the differential ∂_t with the infinite sum in (2.11), by using (2.9). (iii) follows from (i) and (ii) easily.

Proof. By (2.10)

$$\frac{\partial_t^k F(t, x_n + 2^{-n}) - \partial_t^k F(t, x_n)}{F(t, x_n + 2^{-n}) - F(t, x_n)} = \frac{\partial_t^k M_n(t,x)}{M_n(t,x)}.$$

There exist positive integers $\left\{ r(k, (k_j)_j) : \sum_j k_j = k, k_j \ge 0 \right\}$ such that

$$\sum_{k_j \ge 0, \sum_{j=0}^{n-1} k_j = k} r(k, (k_j)_j) = n^k \quad \text{and} \tag{2.17}$$

$$\partial_t^k M_n(t,x) = \sum_{k_j \ge 0, \ \sum_{j=0}^{n-1} k_j = k} C(k, (k_j)_j) \left(\prod_{j=0}^{n-1} \partial_t^{k_j} H_j(t,x) \right).$$

We now compare $\partial_t^{k_j} H_j(t,x)$ with $H_j(t,x)$. Since the number of j such that $k_j > 0$ is less than or equal to k,

$$\left| \prod_{j=0}^{n-1} \frac{\partial_t^{k_j} H_j(t,x)}{H_j(t,x)} \right| = \left| \prod_{j: 0 < k_j \le k} \frac{\partial_t^{k_j} H_j(t,x)}{H_j(t,x)} \right| \le \left(\frac{\max_{0 \le l \le k, j \ge 0, x \in [\alpha(t), \beta(t)]} \left|\partial_t^l H_j(t,x)\right|}{\min_{j \ge 0, x \in [\alpha(t), \beta(t)]} H_j(t,x)} \right)^k.$$

Lemma 2.5 implies for each $l \ge 0$

$$\max_{j \ge 0, x \in [\alpha(t), \beta(t)]} \left|\partial_t^l H_j(t,x)\right| \le C_{1,l}(t) < +\infty.$$

By (2.17)

$$\left| \frac{\partial_t^k M_n(t,x)}{M_n(t,x)} \right| \le \sum_{k_j \ge 0, \ \sum_{j=0}^{n-1} k_j = k} r(k, (k_j)_j) C_{4,k}(t) = C_{4,k}(t) n^k, \tag{2.18}$$

where $C_{4,k}(t) := \max_{0 \le l \le k} C_{1,l}(t)$. This is continuous with respect to t. Thus we have (i).

By (2.18) and (2.9) there is an open interval (a,b) containing 0 such that

$$\max_{t \in [a,b]} C_{4,k}(t) < +\infty, \quad \max_{t \in [a,b]} p_{\max}(t) < 1 \quad \text{and}$$

$$\sum_n \max_{t \in [a,b]} \left| \partial_t^k F(t, x_{n+1}) - \partial_t^k F(t, x_n) \right| \le \max_{t \in [a,b]} C_{4,k}(t) \cdot \sum_{n \ge 0} n^k \left(\max_{t \in [a,b]} p_{\max}(t) \right)^n. \tag{2.19}$$

Recall (2.11). Thus we have (ii).

(2.19) implies $|f_k(x) - f_k(y)| = \lim_{n \to \infty} |f_k(x_n) - f_k(y_n)|$. This and (2.14) imply (2.15). The continuity of $\partial_t^k F(0,x)$ with respect to x follows from (2.15) and the continuity of $F(x)$. Thus we have (iii). □

Hereafter, if $t = 0$ we often omit t and write $F(x) = F(0,x)$ and $\widetilde{F}(x) = \widetilde{F}(0,x)$.

3. Hausdorff dimension

Theorem 3.1. *For any $k \ge 1$, the Hausdorff dimension of the graph of f_k is 1.*

This extends [1, Corollary 4.2] and is applicable to the framework in [3, Section 5]. If $f_1 = T$, this follows from [9]. For proof we will choose a "good" family of coverings of the graph of f_k and then show $\dim_H\{(x, f_k(x)) : x \in [0,1]\} \le s$ for any $s > 1$. The key point is using the simple fact that F is the distribution function of μ_0. Our proof is different from [9] and [1] and seems simpler than them because we do not need to investigate strength of continuity of f_k such as (2.16) and the Hölder exponent. As we will see in Theorem 5.7 (ii) later f_k may not be η-Hölder continuous if $\eta < 1$ is sufficiently close to 1.

Proof. Hereafter, "$\dim_H$" denotes the Hausdorff dimension and "diam" denotes the diameter. It is easy to see $\dim_H\{(x, f_k(x)) : x \in [0,1]\} \ge 1$. We now show $\dim_H\{(x, f_k(x)) : x \in [0,1]\} \le s$ for any $s > 1$. Let

$$O\left(f_k, n, l\right) := \max_{x \in [(l-1)/2^n, l/2^n]} \left| f_k(x) - f_k\left(\frac{l-1}{2^n}\right) \right| \text{ and}$$

$$R(k; n, l) := \left[\frac{l-1}{2^n}, \frac{l}{2^n}\right] \times \left[f_k\left(\frac{l-1}{2^n}\right) - O\left(f_k, n, l\right), f_k\left(\frac{l-1}{2^n}\right) + O\left(f_k, n, l\right)\right].$$

Then $\cup_{l=1}^{2^n} R(k; n, l)$ covers the graph of f_k and

$$\operatorname{diam}\left(R(k; n, l)\right) = \left(4^{-n} + 4O\left(f_k, n, l\right)^2\right)^{1/2}.$$

If $s > 1$,

$$\left(4^{-n} + 4O\left(f_k, n, l\right)^2\right)^{s/2} \le \left(2^{-n} + 2O\left(f_k, n, l\right)\right)^s \le 2^{s-1}\left(2^{-sn} + 2^s O\left(f_k, n, l\right)^s\right).$$

Therefore it suffices to show that

$$\lim_{n \to \infty} \sum_{l=1}^{2^n} O\left(f_k, n, l\right)^s = 0. \tag{3.1}$$

By (2.15)

$$O\left(f_k, n, l\right) \le C_k \max_{x \in [(l-1)/2^n, l/2^n]} \left(-\log_2 \left| x - \frac{l-1}{2^n} \right|\right)^k \left(F(x) - F\left(\frac{l-1}{2^n}\right)\right).$$

Using this and

$$\sum_{l=1}^{2^n} F\left(\frac{l}{2^n}\right) - F\left(\frac{l-1}{2^n}\right) = 1,$$

$$\sum_{l=1}^{2^n} O\left(f_k, n, l\right)^s \le C_k^s \max_{x,y \in [0,1], 0 < |x-y| \le 2^{-n}} \left(-\log_2 |x-y|\right)^{sk} |F(x) - F(y)|^{s-1}.$$

Let $z < w$ and $n = n_{z,w}$ be the smallest number n such that $z \le (k-1)/2^n < k/2^n \le w$ for some k. Then $z \ge \min\{0, (k-3)/2^n\}$ and $w \le \max\{1, (k+2)/2^n\}$. By (2.9), $p_{\max}(0) < 1$ and $\max_{v \in (0,1)} \mu_0([v_n, v_n + 2^{-n})) \le p_{\max}(0)^n$. Hence

$$F(y) - F(x) \le F\left(\max\{1, (k+2)/2^n\}\right) - F\left(\min\{0, (k-3)/2^n\}\right) \le 5p_{\max}(0)^n.$$

Hence

$$|F(z) - F(w)| \le 5|z-w|^c, \; z, w \in [0,1], \quad \text{for } c = -\log_2 p_{\max}(0) > 0.$$

Using this and $s > 1$,

$$\lim_{n \to \infty} \max_{x,y \in [0,1], 0 < |x-y| \le 2^{-n}} \left(-\log_2 |x-y|\right)^{sk} |F(x) - F(y)|^{s-1} = 0.$$

Thus we have (3.1). $\square$

4. Local Hölder continuity at almost every points

Theorem 4.1. *There is $c \geq 1$ such that for any $k \geq 0$ there is $C'_k < +\infty$ such that*

$$\limsup_{h \to 0} \frac{|f_k(x+h) - f_k(x)|}{|h|^c} \leq C'_k \quad \textit{Lebesgue-a.e.}x.$$

If μ_0 is singular, $c > 1$ and $C'_k = 0$ for any k. If μ_0 is absolutely continuous, $c = 1$.

This is more general than [3, Theorems 12 and 13] which investigates the case $(b_1(t), c_0(t), c_1(t)) = (t+a, 0, 0)$, $a \neq 1/2$ only. Our approach is partly similar to the proof of [3, Theorem 12] but seems more general and clearer than it. The key point is showing the following : (1) Giving a nice upper bound for $|F(x) - F(y)|$ in terms of $M_m(0, x)$ by (2.11) and (2.9). (2) $M_m(0, x)$ decays rapidly by (4.2) below. (3) Giving a nice lower bound for $|x - y|$ by assuming x is a normal number as the proof of [3, Theorem 12].

Let

$$\{m_1(z) < m_2(z) < \cdots\} := \{i \geq 1 : X_i(z) = 1\}, \quad z \in [0, 1). \tag{4.1}$$

$$n(x, y) := \min\{n : m_k(x) = m_k(y) \text{ for any } k \leq n\}, \quad x, y \in (0, 1) \setminus D \text{ with } x \neq y.$$

In a manner similar to the proof of [10, Theorem 1.2][7], there is a constant $c \geq 1$ such that

$$\liminf_{n \to \infty} \frac{-\log_2 M_n(0, x)}{n} \geq c \quad \text{Lebesgue-a.e.}x. \tag{4.2}$$

If μ_0 is singular, $c > 1$. If μ_0 is absolutely continuous, $c = 1$.

Proof. This assertion is trivial if μ_0 is absolutely continuous. Assume μ_0 is singular and x is a normal number. Using (2.15), it suffices to show

$$\lim_{h \to 0} \frac{|F(x+h) - F(x)|}{|h|^c} = 0 \quad \text{Lebesgue-a.e.}x \text{ for some } c > 1. \tag{4.3}$$

Let $y \in (0, 1) \setminus (D \cup \{x\})$ and Let $m := m_{n(x,y)}(x)$. Then $X_k(x) = X_k(y)$ and $M_k(0, x) = M_k(0, y)$ for any $k \leq m_{n(x,y)}(x)$. By (2.11) and (2.9),

$$|F(x) - F(y)| \leq \sum_{i \geq m} |M_i(0, x) - M_i(0, y)| \leq C M_m(0, x). \tag{4.4}$$

Here C denotes a constant independent from x, y.

We now give a lower bound of $|x - y|$ in terms of $n(x, y)$. If $x > y$, $m_{n(x,y)+1}(x) < m_{n(x,y)+1}(y)$ and hence

$$x - y \geq 2^{-m_{n(x,y)+2}(x)}.$$

If $x < y$, $m_{n(x,y)+1}(x) > m_{n(x,y)+1}(y)$ and hence

$$y - x \geq 2^{-m_{n(x,y)+1}(x)} - \sum_{j \geq n(x,y)+2} 2^{-m_j(x)}.$$

Since x is normal,

$$|x - y| \geq 2^{-m_{n(x,y)+2}(x) \cdot (1 + o(1))}, \ y \to x. \tag{4.5}$$

By (4.2) and $\displaystyle\lim_{k \to +\infty} \frac{m_{k+2}(x)}{m_k(x)} = 1$, there is $c > 1$ such that

$$\lim_{k \to \infty} 2^{c \cdot m_{k+2}(x)} M_{m_k(x)}(0, x) = 0 \tag{4.6}$$

holds for Lebesgue-a.e. normal number x. We also have $\lim_{y \to x} n(x, y) = +\infty$. Now (4.3) follows from (4.4), (4.5) and (4.6). □

[7](4.2) is a statement for the *Lebesgue measure*. Hence we need to alter the arguments in the proofs of [10, Lemma 2.3 (2) and Lemma 3.3] slightly. Since the alteration is easy we omit the details.

5. Asymptotics of f_k around dyadic rationals

5.1. **Lemmas.** Recall the definition of g_i, P_i and H_i in Definition 2.2. Then

$$\partial_t P_i(t,x) = \frac{b_1'(t)(g_i(t,x)+1) + b_1(t)(1-b_1(t))\partial_t g_i(t,x)}{(b_1(t)g_i(t,x)+1)^2}. \tag{5.1}$$

Let

$$D_n := \left\{ \frac{k}{2^n} : 1 \le k \le 2^n - 1 \right\}, \quad n \ge 1 \quad \text{and} \quad D_0 := \emptyset.$$

Lemma 5.1. (i)

$$\lim_{i\to\infty} \sup_{y \in D_k \setminus D_{k-1}, k \ge 1} |H_{i+k}(0,y) - b_1(1+c_0)| = 0. \tag{5.2}$$

This also holds if we substitute $\widetilde{H}_{i+k}, \widetilde{b}_1$ and $\widetilde{c}_0$ for $H_{i+k}, b_1,$ and c_0.
(ii) *If $x \in D$,*

$$\lim_{n\to\infty} \frac{\partial_t H_n(0,x)}{H_n(0,x)} = \frac{b_1'(0)}{b_1} + \frac{c_0'(0)}{1+c_0} \tag{5.3}$$

Convergences (5.2) and (5.3) are exponentially fast.

Proof. Recall the definition of G_i in (2.7). Let $G_{0,i}$ be the i-th composition of $G_0(0,\cdot)$. Since the Lipschitz constant of $G_0(0,\cdot)$ on $[\alpha,\beta]$ is strictly smaller than 1,

$$\lim_{i\to\infty} \sup_{z\in[\alpha,\beta]} \left| G_{0,i}(z) - \frac{c_0}{1-b_1(1+c_0)} \right| = 0. \tag{5.4}$$

This convergence is exponentially-fast. If $y \in D_k \setminus D_{k-1}$, $H_{i+k}(0,y) = p_0\left(0, G_{0,i}(g_k(0,y))\right)$. Hence (5.2) holds and the convergence is exponentially fast. Since the Lipschitz constant of $\widetilde{G}_0(0,\cdot)$ on $\left[\widetilde{\alpha},\widetilde{\beta}\right]$ strictly smaller than 1, (5.2) holds for $\widetilde{H}_{i+k}, \widetilde{b}_1$ and $\widetilde{c}_0$. Thus we have (i).

We have

$$\partial_t g_i(t,x) = \partial_t G_{X_i(x)}\left(t, g_{i-1}(t,x)\right) + \partial_y G_{X_i(x)}\left(t, g_{i-1}(t,x)\right) \partial_t g_{i-1}(t,x). \tag{5.5}$$

Note $X_n(x) = 0$ for large n. By (5.4) and (5.5)

$$\lim_{n\to\infty} g_n(0,x) = \frac{c_0}{1-b_1(c_0+1)} \quad \text{exponentially fast and}$$

$$\lim_{n\to\infty} \partial_t g_n(0,x) = \frac{\partial_t G_0\left(0, \frac{c_0}{1-b_1(c_0+1)}\right)}{1-\partial_y G_0\left(0, \frac{c_0}{1-b_1(c_0+1)}\right)} \quad \text{exponentially fast.}$$

Using these convergences, (5.1) and (5.2), we have (ii). □

5.2. **Non-degenerate condition.** If all of $b_1(t), c_0(t)$ and $c_1(t)$ are constant, $f_k(x) = 0$ for *any* $x \in [0,1]$ and $k \ge 1$. In this case the estimate in (2.15) is not best. We now introduce a "non-degenerate" condition for the curves and consider the estimate in (2.15) is best or not under the condition.

Definition 5.2 (A non-degenerate condition)**.** We say *(ND) holds* if

$$b_1'(0)(\alpha+1) + b_1(1-b_1)\min\{0,\delta_0,\delta_1\} > 0 \text{ or} \tag{5.6}$$

$$(\widetilde{b}_1)'(0)(\widetilde{\alpha}+1) + \widetilde{b}_1(1-\widetilde{b}_1)\max\{0,\widetilde{\delta}_0,\widetilde{\delta}_1\} < 0, \tag{5.7}$$

$$\text{where} \quad \delta_i := \min_{y\in[\alpha,\beta]} \frac{\partial_t G_i(0,y)}{1-\partial_y G_i(0,y)}, \; \widetilde{\delta}_i := \max_{y\in[\widetilde{\alpha},\widetilde{\beta}]} \frac{\partial_t \widetilde{G}_i(0,y)}{1-\partial_y \widetilde{G}_i(0,y)}, \; i = 0,1.$$

Recall (2.7) for the definitions of G_i and $\widetilde{G}_i$. Both δ_0 and $\widetilde{\delta}_0$ are well-defined. On the other hand either δ_1 or $\widetilde{\delta}_1$ is well-defined.

By this condition the derivative of $F(t, x_n + 2^{-n}) - F(t, x_n)$ with respect to t is positive at $t = 0$. In particular f_k is *not* a constant. See Lemma 5.3 for details. If $\gamma(t)$ is a smooth curve with $\gamma(0) = 0$ and $\gamma'(0) > 0$, (ND) holds for $(b_1(t), c_0(t), c_1(t))$ if and only if it also holds for $(b_1(\gamma(t)), c_0(\gamma(t)), c_1(\gamma(t)))$.

This condition is somewhat complex. However (ND) holds for T and its generalizations in [1], [2], [3] etc. If $c_1(t) = c_2(t) = 0$ for any t, $\delta_0(t) = \delta_1(t) = \widetilde{\delta}_0(t) = \widetilde{\delta}_1(t) = 0$ and hence (ND) holds if and only if $b_1'(0) > 0$. Hereafter we will not use (ND) explicitly. Instead the following will be used.

Lemma 5.3. *If (ND) holds,*

$$\inf_{n \geq 0, x \in [0,1]} \partial_t P_n(0, x) > 0. \tag{5.8}$$

Proof. First assume (5.6). Recall (5.5). If $\eta \leq \min\{\delta_0, \delta_1\}$ and $\partial_t g_{i-1}(0, x) \geq \eta$ then

$$\partial_t g_i(0, x) \geq \eta.$$

Since $\partial_t g_0(0, x) = 0$,

$$\partial_t g_i(0, x) \geq \min\{0, \delta_0, \delta_1\}, i \geq 0.$$

Using (2.8), (5.1) and (5.6), we have (5.8).

Second assume (5.7). Recall (2.4). Then

$$\widetilde{P}_k(t, x) = 1 - P_k(t, 1 - x), \ x \in (0, 1) \setminus D_k, \ k \geq 1.$$

$$\sup_{n \geq 0, x \in [0,1)} \partial_t \widetilde{P}_n(0, x) < 0.$$

(5.8) follows from these claims. □

5.3. Comparing $\Delta_k F(x, x+h)$ with $(\log_2(1/|h|))^k$ at dyadic rationals.

Theorem 5.4. *For any $k \geq 0$ and any $x \in D$,*

$$\lim_{h \to 0, h > 0} \frac{\Delta_k F(x, x+h)}{(\log_2(1/|h|))^k} = \left(\frac{b_1'(0)}{b_1} + \frac{c_0'(0)}{c_0 + 1}\right)^k. \tag{5.9}$$

$$\lim_{h \to 0, h < 0} \frac{\Delta_k F(x, x+h)}{(\log_2(1/|h|))^k} = \left(-\frac{b_1'(0)}{b_1} - \frac{c_1'(0)}{c_1 + 1}\right)^k. \tag{5.10}$$

If (ND) holds, $b_1'(0)/b_1 + c_i'(0)/(c_i + 1)$, $i = 0, 1$, above are positive and hence we can not replace $(-\log_2|x - y|)^k$ with smaller functions in (2.15). This extends Krüppel [8, Proposition 3.2]. [1, Theorem 4.1] follows from this and (2.15).

For proof first consider the asymptotic of $\Delta_k F(x, x + 2^{-n})$ as $n \to \infty$ as in (5.14) below. We will show this by induction on k and Lemma 5.1 (ii). Then replace "2^{-n}" in $\Delta_k F(x, x + 2^{-n})$ with $h > 0$.

Definition 5.5. Let

$$Z_{k,n}(x) := \Delta_k F(x_n, x_n + 2^{-n}), \ x \in [0, 1), n \geq 1, k \geq 0. \tag{5.11}$$

Define $\widetilde{Z}_{k,n}$ by substituting $\widetilde{F}$ for F.

Proposition 5.6. *For any $k \geq 0$ and any $x \in D$,*

$$\lim_{n \to \infty} \frac{\Delta_k F(x, x + 2^{-n})}{n^k} = \left(\frac{b_1'(0)}{b_1} + \frac{c_0'(0)}{c_0 + 1}\right)^k. \tag{5.12}$$

$$\lim_{n \to \infty} \frac{\Delta_k F(x, x - 2^{-n})}{n^k} = \left(-\frac{b_1'(0)}{b_1} - \frac{c_1'(0)}{c_1 + 1}\right)^k. \tag{5.13}$$

Proof. Let $x \in D$. Then $x = x_n$ and $P_n(t,x) = H_n(t,x)$ hold for any t and sufficiently large n. We now show

$$\lim_{n\to\infty} \frac{Z_{k,n}(x)}{n^k} = q_1^k \quad \text{where } q_1 := \frac{b_1'(0)}{b_1} + \frac{c_0'(0)}{1+c_0} \tag{5.14}$$

by induction on k. The case $k=0$ follows immediately. Assume (5.14) holds for any $k = 0, 1, \dots, l-1$. Differentiating

$$F(t, x_{n+1} + 2^{-n-1}) - F(t, x_{n+1}) = \left(F(t, x_n + 2^{-n}) - F(t, x_n)\right) H_n(t,x)$$

l times with respect to t at $t=0$,

$$Z_{l,n+1}(x) = Z_{l,n}(x) + l\frac{\partial_t P_n(0,x)}{P_n(0,x)} Z_{l-1,n}(x) + \sum_{i=2}^{l} \binom{l}{i} \frac{\partial_t^i P_n(0,x)}{P_n(0,x)} Z_{l-i,n}(x). \tag{5.15}$$

By (2.15) and (5.11), $Z_{k,n}(x) = O(n^k)$. By (2.9) and Lemma 2.5,

$$\sum_{i=2}^{l} \binom{l}{i} \frac{\partial_t^i P_n(0,x)}{P_n(0,x)} Z_{l-i,n}(x) = O\left(n^{l-2}\right).$$

Using this, (5.3) and the hypothesis of induction,

$$\lim_{n\to\infty} \frac{Z_{l,n+1}(x) - Z_{l,n}(x)}{n^{l-1}} = l \lim_{n\to\infty} \frac{\partial_t P_n(0,x)}{P_n(0,x)} \frac{Z_{l-1,n}(x)}{n^{l-1}} = l q_1^l.$$

Hence (5.14) holds for $k = l$. Thus we have (5.12).

In the same manner as above

$$\lim_{n\to\infty} \frac{\Delta_k F(x - 2^{-n}, x)}{n^k} = \lim_{n\to\infty} \frac{\widetilde{Z}_{k,n}(1-x)}{n^k} = (-q_2)^k \quad \text{where } q_2 := \frac{b_1'(0)}{b_1} + \frac{c_1'(0)}{c_1+1}.$$

Thus we have (5.13). □

We will show Theorem 5.4 using Proposition 5.6 crucially. Roughly, what we need to show is substituting h for 2^{-n} in Proposition 5.6. Recall (4.1) for the definition of $\{m_n(z)\}_n$.

Proof of Theorem 5.4. Let $x \in D$ and $n_0 := \min\{n : x \in D_n\}$. If $m_1(h) > n_0$,

$$\Delta_k F(x, (x+h)_{m_1(h)}) = Z_{k,m_1(h)}(x)$$

and hence

$$\Delta_k F(x, x+h) = Z_{k,m_1(h)}(x)$$
$$+ \sum_{i=2}^{\infty} \frac{F((x+h)_{m_i(h)}) - F((x+h)_{m_{i-1}(h)})}{F(x+h) - F(x)} \left(Z_{k,m_i(h)}((x+h)_{m_{i-1}(h)}) - Z_{k,m_1(h)}(x)\right).$$

By (2.9) and $((x+h)_{m_{i-1}(h)})_{m_1(h)-1} = x$,

$$\frac{F((x+h)_{m_i(h)}) - F((x+h)_{m_{i-1}(h)})}{F(x+h) - F(x)} \le \frac{M_{m_i(h)}(0, (x+h)_{m_{i-1}(h)})}{M_{m_1(h)-1}(0,x)} \le p_{\max}(0)^{m_i(h)-m_1(h)}.$$

Using (5.15), Lemma 2.5 and (2.14), there is a constant $C_k'' < +\infty$ such that

$$\left|Z_{k,m_i(h)}((x+h)_{m_{i-1}(h)}) - Z_{k,m_1(h)-1}(x)\right| \le C_k'' m_1(h)^{k-1} \left(m_i(h) - m_1(h)\right)^k.$$

Therefore

$$\frac{\left|\Delta_k F(x, x+h) - Z_{k,m_1(h)}(x)\right|}{m_1(h)^k} \le C_k'' \frac{1}{m_1(h)} \sum_{n\ge 1} n^k p_{\max}(0)^n.$$

The right hand side goes to 0 as $h \to 0, h > 0$. By this and (5.12) we have (5.9). We can show (5.10) in the same manner by using (5.13). □

The asymptotic of $f_k(x)$ around $x \in D$ are quite different depending on (b_1, c_0, c_1).

Theorem 5.7. *Let $x \in D$. Then*
(i) *If $c_0 < (1-2b_1)/2b_1$ and $c_1 > 1-2b_1$, there is $c > 1$ such that*

$$\lim_{h \to 0} \frac{|f_k(x+h) - f_k(x)|}{|h|^c} = 0. \tag{5.16}$$

(ii) *Assume (ND) holds. If $c_0 \ge (1-2b_1)/2b_1$ or $c_1 \le 1-2b_1$, there is $c \le 1$ such that*

$$\limsup_{h \to 0} \frac{|f_k(x+h) - f_k(x)|}{|h|^c (\log_2(1/|h|))^k} = +\infty. \tag{5.17}$$

If μ_0 is singular, $c < 1$. If μ_0 is absolutely continuous, $c = 1$.

(i) is similar to Theorem 4.1 and consistent with (2.2) and (2.6). An example of a graph of f_1 satisfying $c_0 < (1-2b_1)/2b_1$ and $c_1 > 1-2b_1$ is given in Figure 1 below. We will show (i) in a manner similar to the proof of Theorem 4.1. The key point is showing $M_m(0,x)$ decays rapidly. We will show it by Lemma 5.1 (i), which plays the same role as (4.2) in the proof of Theorem 4.1. If μ_0 is absolutely continuous or $c_0 = c_1 = 0$, $c_0 \ge (1-2b_1)/2b_1$ or $c_1 \le 1-2b_1$. For the proof of (ii), by Theorem 5.4, it suffices to show $|F(x+h) - F(x)| \ge c|h|^c$. We will show it by Lemma 5.1 (i).

Proof. Let $x \in D$. By Lemma 5.1 (i) and $b_1(c_0+1) < 1/2$,

$$\lim_{m \to \infty} 2^{cm} m^k M_m(0,x) = 0 \text{ for some } c > 1.$$

Using this, (2.14) and (4.4),

$$\lim_{h \to 0, h > 0} \frac{|f_k(x+h) - f_k(x)|}{|h|^c} = 0. \tag{5.18}$$

Since $c_1 > 1-2b_1$, $\widetilde{b}_1(\widetilde{c}_0+1) < 1/2$ and (5.18) holds also for $\partial_t^k \widetilde{F}(0,x)$. By (2.4)

$$\partial_t^k \widetilde{F}(t,x) = -\partial_t^k F(t, 1-x), \quad x \in (0,1),\ k \ge 1.$$

Therefore

$$\lim_{h \to 0, h > 0} \frac{|f_k(x) - f_k(x-h)|}{|h|^c} = 0. \tag{5.19}$$

(5.18) and (5.19) imply (5.16).

We now show (ii). Assume $c_0 \ge (1-2b_1)/2b_1$. It is equivalent to $b_1(c_0+1) \ge 1/2$. By Lemma 5.1 (i), for some $c \le 1$ which does not depend on x,

$$\liminf_{n \to \infty} 2^{c \cdot n} (F(x+2^{-n}) - F(x)) > 0. \tag{5.20}$$

Assume $c_1 \le 1-2b_1$. Then $\widetilde{c}_0 \ge (1-2\widetilde{b}_1)/2\widetilde{b}_1$. Therefore (5.20) holds for $\widetilde{F}$. Hence for some $c \le 1$ which does not depend on x,

$$\liminf_{n \to \infty} 2^{c \cdot n} (F(x) - F(x-2^{-n})) = \liminf_{n \to \infty} 2^{c \cdot n} (\widetilde{F}(1-x+2^{-n}) - \widetilde{F}(1-x)) > 0. \tag{5.21}$$

Since either (5.20) or (5.21) holds,

$$\limsup_{h \to 0} \frac{|F(x+h) - F(x)|}{|h|^c} > 0.$$

If μ_0 is singular, $c < 1$. If it is absolutely continuous, $c = 1$. Using this, Lemma 5.3 and Theorem 5.4, we have (5.17). □

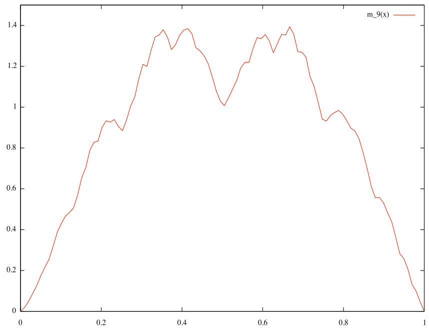

FIGURE 1. Graph of f_1 for $(b_1(t), c_0(t), c_1(t)) = \left(t + \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}\right)$.

6. Results for two special cases

We say *(L) holds* if $(b_1(t), c_0(t), c_1(t)) = (t + a, 0, 0)$ for some $a \in (0, 1)$. *In this section we always assume (ND) holds and either $k = 1$ or (L) holds.* Recall Definition 5.2 and Lemma 5.3 for (ND).

Let

$$Y_i(x) := \frac{\partial_t H_{i-1}(0, x)}{H_{i-1}(0, x)}, \; i \ge 1, \text{ and } Y_0(x) := 0.$$

$Y_i(x) > 0$ if and only if $X_i(x) = 0$. If (L) holds,

$$Y_i(x) = \frac{1}{a} 1_{\{X_i(x)=0\}}(x) + \frac{1}{1-a} 1_{\{X_1(x)=1\}}(x).$$

Lemmas 2.5 and 5.3 imply

$$0 < \inf_{i \ge 1, x \in (0,1)} |Y_i(x)| \le \sup_{i \ge 1, x \in (0,1)} |Y_i(x)| < +\infty. \tag{6.1}$$

Recall the definition of $Z_{k,n}$ in (5.11). Then

$$Z_{1,n}(x) = \sum_{i=1}^{n} Y_i(x). \tag{6.2}$$

If (L) holds, using (5.15),

$$Z_{k,n+1}(x) - Z_{k,n}(x) = k Y_{n+1}(x) Z_{k-1,n}(x), \quad x \in [0, 1), \quad k \ge 2. \tag{6.3}$$

Let $\mu_0(\cdot|A)$ be the conditional probability of μ_0 given a Borel measurable set A. Denote the expectation with respect to $\mu_0(\cdot|A)$ by $E_A^{\mu_0}$. Let

$$\mathcal{F}_n := \sigma\left(\left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \;\middle|\; 0 \le k \le 2^n - 1\right\}\right), \quad n \ge 0.$$

Then $\{Z_{k,i}\}_{i \ge n}$ is a $\{\mathcal{F}_i\}_i$-martingale[8] with respect to $\mu_0(\cdot|A)$ for $A \in \mathcal{F}_n$. By induction on k

$$Z_{k,n} = k! \sum_{1 \le i_1 < \cdots < i_k \le n} \left(\prod_{j=1}^{k} Y_{i_j}\right) = O(n^k).$$

Lemma 6.1 (Fluctuation of $\{Z_{k,n}\}_n$). *For each $k \ge 1$*

$$\limsup_{m \to +\infty} Z_{k,m}(x) > \liminf_{m \to +\infty} Z_{k,m}(x), \quad x \in (0, 1). \tag{6.4}$$

$$\limsup_{m \to \infty} E_A^{\mu_0}[|Z_{k,m}|] = +\infty, \quad A \in \mathcal{F}_n, n \ge 1. \tag{6.5}$$

[8]See Williams' book [15] for definition.

Proof. The case $k = 1$ of (6.4) follows from (6.1). Assume (L) holds. We now show this by induction on k. Assume that this assertion holds for $k = 1, \dots, l$ and

$$\limsup_{n\to+\infty} Z_{l+1,n}(x) = \liminf_{n\to+\infty} Z_{l+1,n}(x) \quad \text{for some } x.$$

Then (6.3) and (6.1) imply $\lim_{n\to\infty} Z_{l,n}(x) = 0$. This contradicts the assumption of induction. Hence

$$\limsup_{n\to+\infty} Z_{l+1,n}(x) > \liminf_{n\to+\infty} Z_{l+1,n}(x)$$

for *any* x. Thus we have (6.4). Using this, (6.4) and the martingale convergence theorem ([15, Chapter 11]), we have (6.5). □

6.1. **Differentiablity and variation.** For $g : [0,1] \to \mathbb{R}$ and $0 \le a \le b \le 1$, let

$$V(g;[a,b]) := \sup\left\{ \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| \;\middle|\; a = t_0 < t_1 < \cdots < t_n = b \right\}.$$

Theorem 6.2. (i) (*Non-differentiablity for the absolutely continuous case*) *For any* $x \in (0,1)$, $\Delta_k F(x, x+h)$ *does not converge to any real number as* $h \to 0$. *In particular, if* μ_0 *is absolutely continuous,* f_k *is not differentiable at any point in* $(0,1)$.
(ii) (*Variation*) $V(f_k;[a,b]) = +\infty$ *holds for any* $[a,b] \subset [0,1]$, $a < b$.

(i) is an extension of [1, Theorem 5.1]. A problem of this kind was also considered by [14]. Our proof of (i) is somewhat similar to Billingsley [4] and [1, Theorem 5.1]. The key is a fluctuation of $\{Z_{k,n}\}_n$ in (6.4). It seems natural to consider whether f_k is of bounded variation. To our knowledge variations of f_k have not been considered. The key of proof of (ii) is showing, by using (6.5), the expectation of $|Z_{k,m}|$ under μ_0 on an interval diverges to infinity.

Proof. If $x \in D$, the assertion follows from Theorem 5.4 and the condition (ND). Assume $x \notin D$. It is easy to see that for any $k, n \ge 1$

$$\begin{aligned} \min\left\{\Delta_k F(x, x_n), \Delta_k F(x, x_n + 2^{-n})\right\} &\le Z_{k,n}(x) \\ &\le \max\left\{\Delta_k F(x, x_n), \Delta_k F(x, x_n + 2^{-n})\right\} \end{aligned} \tag{6.6}$$

By (6.4) $Z_{k,n}(x)$ does not converge to any real number. Therefore if $(Z_{k,n}(x))_n$ diverges as $n \to +\infty$,

$$\limsup_{h\to 0} |\Delta_k F(x, x+h)| = +\infty.$$

If $(Z_{k,n}(x))_n$ fluctuates as $n \to +\infty$,

$$\begin{aligned} \limsup_{h\to 0} \Delta_k F(x, x+h) - \liminf_{h\to 0} \Delta_k F(x, x+h) &\ge \limsup_{n\to+\infty} Z_{k,n}(x) - \liminf_{n\to+\infty} Z_{k,n}(x) \\ &\ge c \end{aligned}$$

for some $c = c(x) > 0$. These imply (i). By (5.11),

$$\sum_{l=2^{m-n}(j-1)+1}^{2^{m-n}j} \left| f_k\left(\frac{l}{2^m}\right) - f_k\left(\frac{l-1}{2^m}\right) \right| = E^{\mu_0}_{[(j-1)/2^n, j/2^n)}\left[|Z_{k,m}|\right], \; m > n.$$

(ii) follows from this and (6.5). □

6.2. **MTNI.**

Theorem 6.3 (MTNI). *For some* $c \in [0,1]$ *the following hold :*
(i)

$$\limsup_{h\to 0} \frac{|f_k(x+h) - f_k(x)|}{|h|^c} = +\infty \quad \mu_0\text{-a.s.}x. \tag{6.7}$$

(ii) *For any open interval* J

$$\sup_{x,y\in J, x>y} \frac{f_k(x) - f_k(y)}{(x-y)^c} = +\infty \quad \textit{and} \quad \inf_{x,y\in J, x>y} \frac{f_k(x) - f_k(y)}{(x-y)^c} = -\infty$$

If μ_0 is singular, $c < 1$. If μ_0 is absolutely continuous, $c = 1$. If (L) holds, c does not depend on k.

(i) corresponds to Theorem 4.1 but here the limit diverges. If μ_0 is singular, the asymptotic of f_k *around Lebesgue-a.e. points* are quite different from the ones *around μ_0-a.s. points*. (ii) extends [3, Theorem 14]. The proof of [3, Theorem 14][9] is omitted in [3]. However the reason that the proof of [3, Proposition 6] is not applied to *even* k is not described in [3]. We will give a proof applied to *all* k together.

For the proofs we first compare $\Delta_k F$ with $Z_{k,n}$ by (6.6) and then estimate $F(x_n + 2^{-n}) - F(x_n)$ by (6.8) below. For (i) we will give a lower bound for $|f_k(x_n + 2^{-n}) - f_k(x_n)|$ in terms of $|Z_{k,n}|$. Remark that $|Z_{k,n}|$ is positive by (6.4). For (ii), by probabilistic techniques we will choose x such that $f_k(x_n + 2^{-n}) - f_k(x_n)$ is "larger" than the positive part of $Z_{k,n}$, roughly speaking.

For $k = 2$ we will give an example of graph of f_2 in Figure 2 below.

Proof. By [10, Lemma 2.3 (2) and Lemma 3.3], there is a constant $c \le 1$ such that

$$\limsup_{n \to \infty} \frac{-\log_2 M_n(0, x)}{n} \le c \;\; \mu_0\text{-a.s.}x. \tag{6.8}$$

If μ_0 is singular, $c < 1$. If μ_0 is absolutely continuous, $c = 1$.

By (6.6) and (6.8)

$$\max\left\{ \frac{|f_k(x) - f_k(x_n)|}{(x - x_n)^c}, \frac{|f_k(x_n + 2^{-n}) - f_k(x)|}{(x_n + 2^{-n} - x)^c} \right\} \ge \frac{1}{2} \left| Z_{k,n}(x) \right|,$$

for large n and μ_0-a.s.$x \in (0,1) \setminus D$. (6.4) implies

$$\limsup_{n \to \infty} |Z_{k,n}(x)| > 0$$

holds for any x. Thus we have (i).

Fix l and n. Denote $E^{\mu_0}_{[(l-1)/2^n, l/2^n)}$ by E. $Z^+_{k,m}$ and $Z^-_{k,m}$ denotes the positive and negative parts of $Z_{k,m}$. $Z^+_{k,m} - Z^-_{k,m} = Z_{k,m}$ and $Z^+_{k,m} + Z^-_{k,m} = |Z_{k,m}|$. Using (6.5) and that $\{|Z_{k,m}|\}_m$ is a submartingale,

$$\lim_{m \to \infty} E[Z^+_{k,m}] + E[Z^-_{k,m}] = \lim_{m \to \infty} E[|Z_{k,m}|] = \sup_m E[|Z_{k,m}|] = +\infty.$$

Since $\{Z_{k,m}\}_{m \ge n}$ is a martingale, $E[Z^+_{k,m}] - E[Z^-_{k,m}] = E[Z_{k,n}]$ for any $m \ge n$. Therefore

$$\lim_{m \to \infty} E[Z^+_{k,m}] = \lim_{m \to \infty} E[Z^-_{k,m}] = +\infty. \tag{6.9}$$

Let

$$A_m := \left\{ x : H_m(0, x) \le 2^{-1-c_1 m} \right\}.$$

By Azuma's inequality ([15, Chapter E.14]) there are constants $c_1 \in [0,1], c_2, c_3 \in (0, +\infty)$ such that for any m, $\mu_0(A_m) \le c_2 \exp(-c_3 m)$. This and (2.14) imply

$$E\left[Z^+_{k,m}, A_m\right] \le C_k m^k \frac{\mu_0(A_m)}{\mu_0\left([(l-1)/2^n, l/2^n)\right)} \to 0, \;\; m \to \infty.$$

If $\left\{ Z^+_{k,m} \ge E\left[Z^+_{k,m}\right]/2 \right\} \subset A_m$ for large m,

$$\limsup_{m \to \infty} E\left[Z^+_{k,m}\right] \le 2 \limsup_{m \to \infty} E\left[Z^+_{k,m}, A_m\right] = 0.$$

This contradicts (6.9). Hence

$$\left\{ Z^+_{k,m} \ge \frac{E[Z^+_{k,m}]}{2} \right\} \cap A^c_m \cap \left[\frac{l-1}{2^n}, \frac{l}{2^n} \right) \ne \emptyset$$

[9]As far as the author sees, the proof of [3, Theorem 14] seems more complex than the one of [3, Proposition 6].

holds for infinitely many m. Using this, (6.6) and (6.8), for c in (6.8)

$$\sup_{x,y \in [(l-1)/2^n, l/2^n], x>y} \frac{f_k(x) - f_k(y)}{(x-y)^c} = +\infty. \tag{6.10}$$

Since $\lim_{m\to\infty} E[Z^-_{k,m}] = +\infty$, there is $c \in [0,1]$ such that

$$\inf_{x,y \in [(l-1)/2^n, l/2^n], x>y} \frac{f_k(x) - f_k(y)}{(x-y)^c} = -\infty. \tag{6.11}$$

(6.10) and (6.11) imply (ii). $\square$

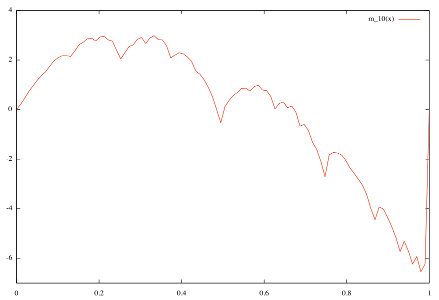

FIGURE 2. Graph of f_2 for $(b_1(t), c_0(t), c_1(t)) = \left(t + \dfrac{1}{3}, 0, 0\right)$.

7. MODULUS OF CONTINUITY

In this section we always assume (ND) holds and $k = 1$. First we will give some notation and lemmas. Second we will compare $\Delta_1 F(x, x+h)$ with $\log_2(1/|h|)$ for $x \notin D$. Finally we will consider a modulus of continuity for $\Delta_1 F(x, x+h)$ at μ_0-a.s.x.

Let

$$l(y,z) := \min\{i \geq 1 : X_i(y) \neq X_i(z)\}, \quad y \neq z.$$

Recall (4.1) for the definition of $m_1(z)$.

Lemma 7.1 ([7, Lemma 2]). *Let $x \notin D$ and $h > 0$. Then*
(i) $\lim_{h\to 0, h>0} l(x, x+h) = +\infty$.
(ii) $l(x, x+h) \leq m_1(h)$.
(iii) $X_i(x) = X_i(x+h)$ *for* $1 \leq i \leq l(x, x+h) - 1$.
(iv) $X_{l(x,x+h)}(x) = 0$ *and* $X_{l(x,x+h)}(x+h) = 1$.
(v) $X_i(x) = 1$ *and* $X_i(x+h) = 0$ *for* $l(x, x+h) < i \leq m_1(h) - 1$.

Define

$$l_x := \min\{j > l(x,x+h) : X_j(x) = 0\} \text{ and } l_{x+h} := \min\{j > l(x,x+h) : X_j(x+h) = 1\}.$$

We have

$$(x+h)_{l_{x+h}-1} = (x+h)_{l(x,x+h)} = x_{l_x - 1} + 2^{-(l_x - 1)}. \tag{7.1}$$

Lemma 7.2 (Key lemma). *Let $x \notin D$ and $h > 0$. Then*

$$\begin{aligned}\Delta_1 F(x, x+h) = {} & \frac{F((x+h)_{l(x,x+h)}) - F(x)}{F(x+h) - F(x)} Z_{1,l_x}(x) \\ & + \frac{F(x+h) - F((x+h)_{l(x,x+h)})}{F(x+h) - F(x)} Z_{1,l_{x+h}}(x+h) + O(1).\end{aligned} \tag{7.2}$$

Proof. By (2.11)

$$f_1(x_{n+1}) - f_1(x_n) = X_{n+1}(x)\left(F(x_n + 2^{-n-1}) - F(x_n)\right)\left(Z_{1,n}(x) + \overline{Y_n}(x)\right),$$

$$\text{where} \quad \overline{Y_n}(x) := -\frac{H_n(0, x_n + 2^{-n-1})}{H_n(0, x_n)} Y_{n+1}(x_n + 2^{-n-1}).$$

$\{\overline{Y_n}\}_n$ are bounded by (2.8) and (6.1). Summing up over n,

$$f_1(x) - f_1(x_k) = (F(x) - F(x_k))Z_{1,k}(x) + J(x,k)$$

$$\text{where} \quad J(x,k) := \sum_{n=k}^{\infty} X_{n+1}(x)\left(F(x_n + 2^{-n-1}) - F(x_n)\right)\left(\sum_{i=k+1}^{n} Y_i(x) + \overline{Y_n}(x)\right).$$

(2.8) and (6.1) imply $J(x,k) = O\left(F(x_k + 2^{-k}) - F(x_k)\right)$. Therefore

$$f_1(x+h) - f_1((x+h)_{l_{x+h}-1}) = (F(x+h) - F((x+h)_{l_{x+h}-1}))\left(Z_{1,l_{x+h}}(x+h) + O(1)\right) \text{ and}$$

$$f_1(x) - f_1(x_{l_x-1}) = (F(x) - F(x_{l_x-1}))Z_{1,l_x-1}(x) + O\left(F(x_{l_x-1} + 2^{-(l_x-1)}) - F(x_{l_x-1})\right).$$

(7.1) implies

$$f_1((x+h)_{l_{x+h}-1}) - f_1(x_{l_x-1}) = \left(F((x+h)_{l_{x+h}-1}) - F(x_{l_x-1})\right) Z_{1,l_x-1}(x).$$

Therefore

$$\begin{aligned} f_1(x+h) - f_1(x) &= \left(F((x+h)_{l_{x+h}-1}) - F(x)\right) Z_{1,l_x}(x) \\ &+ \left(F(x+h) - F((x+h)_{l_{x+h}-1})\right)\left(Z_{1,l_{x+h}}(x+h) + O(1)\right) \\ &+ O\left(F(x_{l_x-1} + 2^{-(l_x-1)}) - F(x_{l_x-1})\right). \end{aligned}$$

Since $F(x_{l_x-1} + 2^{-(l_x-1)}) - F(x_{l_x-1}) = O\left(F(x+h) - F(x)\right)$ we have (7.2). □

7.1. **Modulus of continuity at non-dyadic rationals.** Recall (4.1) for the definition of $\{m_n(z)\}_n$.

Theorem 7.3. *Assume $x \notin D$. Then*

$$\lim_{h\to 0, h>0} \frac{\Delta_1 F(x, x+h)}{\log_2(1/h)} \text{ exists as a real number}$$

if and only if

$$\lim_{n\to\infty} \frac{m_{n+1}(1-x)}{m_n(1-x)} = 1 \text{ and } \lim_{n\to\infty} \frac{Z_{1,n}(x)}{n} \text{ exists.}$$

If they hold,

$$\lim_{h\to 0, h>0} \frac{\Delta_1 F(x, x+h)}{\log_2(1/h)} = \lim_{n\to\infty} \frac{Z_{1,n}(x)}{n}.$$

Considering also the limit from left we have Corollary 7.6, which extends [2, Theorem 5.4]. [2] uses Kôno's expression [7, Lemma 3]. If (L) holds then we may expect a counterpart of [7, Lemma 3]. However if (L) fails then it seems impossible to obtain a counterpart of [7, Lemma 3]. The key point is Lemma 7.2 above, which states $\Delta_1 F(x, x+h)$ is between $Z_{1,l_x}(x)$ and $Z_{1,l_{x+h}}(x+h)$, roughly speaking. In Propositions 7.4 and 7.5 below we will investigate the asymptotic of $\Delta_1 F(x, x+h) - Z_{1,l(x,x+h)}(x)$. These results are different depending on the asymptotic for $\dfrac{m_{n+1}(1-x)}{m_n(1-x)}$. Eliminate parts consisting of the differentials by estimating $Z_{1,l_x}(x) - Z_{1,l(x,x+h)}(x)$ and $Z_{1,l_{x+h}}(x+h) - Z_{1,l(x,x+h)}(x)$. Finally consider quantities expressed by F, x and h only as in (7.3) and (7.4) below.

Proposition 7.4. *If $x \notin D$ and* $\displaystyle\lim_{n\to\infty} \frac{m_{n+1}(1-x)}{m_n(1-x)} = 1$,

$$\lim_{h\to 0, h>0} \frac{\Delta_1 F(x, x+h) - Z_{1,\lfloor \log_2(1/h) \rfloor}(x)}{\log_2(1/h)} = 0.$$

Here $\lfloor x \rfloor$ denotes the maximal integer less than or equal to x.

Proof. First we remark $\lim_{h\to 0, h>0} \frac{l(x,x+h)}{m_1(h)} = 1$ by the assumptions. By (6.1), (6.2) and (7.2) it suffices to show

$$\lim_{h\to 0,h>0} \frac{F((x+h)_{l_{x+h}-1}) - F(x)}{F(x+h)-F(x)} \cdot \frac{l_x - m_1(h)}{m_1(h)} = 0 \text{ and} \tag{7.3}$$

$$\lim_{h\to 0,h>0} \frac{F(x+h) - F((x+h)_{l_{x+h}-1})}{F(x+h)-F(x)} \cdot \frac{l_{x+h} - l(x,x+h) + m_1(h) - l(x,x+h)}{m_1(h)} = 0. \tag{7.4}$$

Using Lemma 7.1 and $\lim_{n\to\infty} \frac{m_{n+1}(1-x)}{m_n(1-x)} = 1$ we have (7.3).

We now show (7.4). We will give an upper bound for

$$(*) := \frac{F(x+h) - F((x+h)_{l_{x+h}-1})}{F(x+h)-F(x)} \times \left(\frac{l_{x+h}}{l(x,x+h)} - 1\right).$$

If $l_{x+h} \le (1+\epsilon) l(x,x+h)$, $(*) \le \epsilon$.

Using (2.9) and

$$M_{l(x,x+h)-1}\left(0,(x+h)_{l_{x+h}-1}\right) = M_{l(x,x+h)-1}\left(0, x_{l_x} + 2^{-l_x}\right),$$

there are constants $0 < c < 1 \le C < +\infty$ such that

$$\begin{aligned}\frac{F(x+h) - F((x+h)_{l_{x+h}-1})}{F(x+h)-F(x)} &\le \frac{M_{l_{x+h}-1}\left(0,(x+h)_{l_{x+h}-1}\right)}{M_{l_x}\left(0, x_{l_x} + 2^{-l_x}\right)} \\ &= C^{l_x - l(x,x+h)} c^{l_{x+h} - l(x,x+h)}\end{aligned}$$

For any $\epsilon > 0$ there is $\delta(\epsilon) > 0$ with $c^{\epsilon/2} < C^{\delta(\epsilon)}$. Therefore if $l_{x+h} \ge (1+\epsilon) l(x,x+h)$ and h is sufficiently small, $l_x \le (1+\delta(\epsilon)) l(x,x+h)$ and

$$\left(\frac{l_{x+h}}{l(x,x+h)} - 1\right)\left(\frac{c^{l_{x+h}/l(x,x+h)-1}}{C^{l_x/l(x,x+h)-1}}\right)^{l(x,x+h)} \le \frac{(l_{x+h} - l(x,x+h)) c^{(l_{x+h}-l(x,x+h))/2}}{l(x,x+h)}.$$

Hence $(*) \le \epsilon$. Thus we have (7.4). □

Proposition 7.5. *If $x \notin D$ and $\limsup_{n\to\infty} \frac{m_{n+1}(1-x)}{m_n(1-x)} > 1$,*

$$\frac{\Delta_1 F(x,x+h)}{\log_2(1/h)} \text{ fluctuates if } h \to 0, h > 0.$$

Proof. Let $\delta > 0$ and $(n(j))_j$ be an increasing sequence satisfying

$$m_{n(j)}(1-x) \ge m_{n(j)-1}(1-x)(1+\delta).$$

Assume $b_1(1+c_0)(1+c_1) \ge 1 - b_1$. Let

$$m(1,j) := m_{n(j)}(1-x) - 2 \text{ and } m(2,j) := m_{n(j)}(1-x).$$

Then

$$F(x + 2^{-m(1,j)}) - F(x) \le M_{m(1,j)}(0, x_{m(1,j)}) + M_{m(1,j)}\left(0, x_{m(1,j)} + 2^{-m(1,j)}\right), \tag{7.5}$$

$$F(x + 2^{-m(1,j)}) - F\left((x + 2^{-m(1,j)})_{m(1,j)}\right) \ge M_{m(1,j)}\left(0, x_{m(1,j)} + 2^{-m(1,j)}\right) \text{ and} \tag{7.6}$$

$$\frac{M_{m(1,j)}\left(0, x_{m(1,j)} + 2^{-m(1,j)}\right)}{M_{m(1,j)}(0, x_{m(1,j)})} = \prod_{i=m_{n(j)-1}(1-x)}^{m_{n(j)}(1-x)-1} \frac{H_i\left(0, x_{m(1,j)} + 2^{-m(1,j)}\right)}{H_i(0, x_{m(1,j)})}. \tag{7.7}$$

By Lemma 5.2 (i) and $m_{n(j)}(1-x) - m_{n(j)-1}(1-x) \ge (1+\delta) j$,

$$\lim_{j\to\infty} \prod_{i=m_{n(j)-1}(1-x)}^{m_{n(j)}(1-x)-1} \frac{H_i\left(0, x_{m(1,j)} + 2^{-m(1,j)}\right)}{b_1(c_0+1)} = 1. \tag{7.8}$$

$$\lim_{j\to\infty} \prod_{i=m_{n(j)-1}(1-x)}^{m_{n(j)}(1-x)-1} \frac{H_i(0, x_{m(1,j)})(1+c_0)}{1-b_1} = 1. \tag{7.9}$$

Using (7.5), (7.6), (7.7), (7.8), (7.9) and $b_1(1+c_0)(1+c_1) \geq 1-b_1$,

$$\liminf_{j\to\infty} \frac{F(x+2^{-m(1,j)}) - F\left((x+2^{-m(1,j)})_{m(1,j)}\right)}{F(x+2^{-m(1,j)}) - F(x)} \geq \frac{1}{2}. \tag{7.10}$$

Since

$$m(1,j) - l(x, x+2^{-m(1,j)}) = m_{n(j)}(1-x) - m_{n(j)-1}(1-x) - 2,$$

$$\liminf_{j\to\infty} \frac{m(1,j) - l(x, x+2^{-m(1,j)})}{m(1,j)} > 0.$$

Using this, (7.10) and (7.2),

$$\liminf_{j\to\infty} \frac{\Delta_1 F(x, x+2^{-m(1,j)}) - Z_{1,m(1,j)}(x)}{m(1,j)} > 0. \tag{7.11}$$

Recall $m(2,j) = l(x, x+2^{-m(2,j)})$ and (7.2). Considering the cases $X_{m(2,j)}(x) = 0$ and $X_{m(2,j)}(x) = 1$ respectively,

$$\limsup_{j\to\infty} \frac{\Delta_1 F(x, x+2^{-m(2,j)}) - Z_{1,m(2,j)}(x)}{m(2,j)} \leq 0. \tag{7.12}$$

By (6.1),

$$\lim_{j\to\infty} \frac{Z_{1,m(1,j)}(x)}{m(1,j)} - \frac{Z_{1,m(2,j)}(x)}{m(2,j)} = 0. \tag{7.13}$$

Using (7.11), (7.12) and (7.13), we have the assertion.

If $b_1(1+c_0)(1+c_1) < 1-b_1$, by Lemma 5.1 (ii) there are $c', c'' > 0$ and $\delta_1, \delta_2 \in (0, \delta)$ such that for large j

$$\Delta_1 F\left(x, x+2^{-(1+\delta_1)n(j-1)}\right) \geq Z_{1,n(j-1)}(x) + c'\delta_1 n(j-1) \text{ and}$$

$$\Delta_1 F\left(x, x+2^{-(1+\delta_2)n(j-1)}\right) \leq Z_{1,n(j-1)}(x) - c''\delta_2 n(j-1).$$

For large j,

$$\frac{\Delta_1 F\left(x, x+2^{-(1+\delta_1)n(j-1)}\right)}{(1+\delta_1)n(j-1)} - \frac{\Delta_1 F(x, x+2^{-n(j-1)})}{n(j-1)} \geq \frac{c'}{2}\delta_1 \text{ if } Z_{1,n(j-1)}(x) < 0.$$

$$\frac{\Delta_1 F\left(x, x+2^{-(1+\delta_2)n(j-1)}\right)}{(1+\delta_2)n(j-1)} - \frac{\Delta_1 F(x, x+2^{-n(j-1)})}{n(j-1)} \leq -\frac{c''}{2}\delta_2 \text{ if } Z_{1,n(j-1)}(x) > 0.$$

Thus we have the assertion. □

Theorem 7.3 follows from Propositions 7.4 and 7.5.

Note

$$\Delta_1 F(x, x+h) = \Delta_1 \widetilde{F}(1-x, 1-x-h) \text{ and } \widetilde{Z}_{1,n}(1-x) = Z_{1,n}(x), \quad x \notin D.$$

We can consider $\displaystyle\lim_{h\to 0, h<0} \frac{\Delta_1 F(x, x+h)}{\log_2(1/|h|)}$ in the same manner. We have

Corollary 7.6. *Assume $x \notin D$. Then*

$$\lim_{h\to 0} \frac{\Delta_1 F(x, x+h)}{\log_2(1/|h|)} \textit{ exists}$$

if and only if

$$\lim_{n\to\infty} \frac{m_{n+1}(1-x)}{m_n(1-x)} = \lim_{n\to\infty} \frac{m_{n+1}(x)}{m_n(x)} = 1 \textit{ and } \lim_{n\to\infty} \frac{Z_{1,n}(x)}{n} \textit{ exists.}$$

7.2. Modulus of continuity at μ_0-a.s. points.

Theorem 7.7. *There are two constants $0 < c \le C < +\infty$ such that the following hold for μ_0-a.s. x :*

$$c \le \limsup_{h \to 0} \frac{\Delta_1 F(x, x+h)}{(\log_2(1/|h|) \log\log\log_2(1/|h|))^{1/2}} \le C. \tag{7.14}$$

$$-C \le \liminf_{h \to 0} \frac{\Delta_1 F(x, x+h)}{(\log_2(1/|h|) \log\log\log_2(1/|h|))^{1/2}} \le -c. \tag{7.15}$$

By this we can improve (6.7) for $k = 1$ as follows[10] :

$$\limsup_{h \to 0} \frac{f_1(x+h) - f_1(x)}{|h|^c} = +\infty = -\liminf_{h \to 0} \frac{f_1(x+h) - f_1(x)}{|h|^c} \quad \mu_0\text{-a.s.}x.$$

As we will see in Corollaries 7.9 and 7.10, for some special choices of $(b_1(t), c_0(t), c_1(t))$, we can improve (7.14) and (7.15).

The key tools of the lower bound for (7.14) and the upper bound for (7.15) are (6.6) and Stout's law of the iterated logarithm (LIL) below. Recall (6.2). Apply Stout's law of the iterated logarithm (LIL) below to $\{Y_i\}_i$. The key tool of the upper bound for (7.14) and the lower bound for (7.15) is Lemma 7.2 above, which states $\Delta_1 F(x, x+h)$ is between $Z_{1,l_x}(x)$ and $Z_{1,l_{x+h}}(x+h)$, roughly. Estimate parts consisting of the differentials by estimating $Z_{1,l_x}(x) - Z_{1,l(x,x+h)}(x)$ and $Z_{1,l_{x+h}}(x+h) - Z_{1,l(x,x+h)}(x)$. Now apply Stout's LIL to $Z_{1,l(x,x+h)}(x)$ and these differences.

Lemma 7.8 (Stout's LIL for martingales [13])**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{S_n, \mathcal{F}_n\}_{n \ge 0}$ be a martingale on it. Let $I_n := \sum_{i=1}^n E[(S_i - S_{i-1})^2 | \mathcal{F}_{i-1}]$ where we denote the expectation with respect to P by E. Assume there are constants $0 < c \le C < +\infty$ such that $c \le |S_i - S_{i-1}| \le C$ μ_0-a.s. for any $i \ge 1$. Then*

$$\limsup_{n \to \infty} \frac{S_n}{(I_n \log\log I_n)^{1/2}} = \sqrt{2} = -\liminf_{n \to \infty} \frac{S_n}{(I_n \log\log I_n)^{1/2}} \quad P\text{-a.s.}$$

Proof of Theorem 7.7. First we show the lower bound for (7.14) and the upper bound for (7.15). Recall (6.6). Applying Lemma 7.8 to $\{Y_i\}_i$, there is $c > 0$ such that the following hold μ_0-a.s.x :

$$\limsup_{n \to +\infty} \frac{\max\{\Delta_1 F(x, x_n), \Delta_1 F(x, x_n + 2^{-n})\}}{(n \log\log n)^{1/2}} \ge c. \tag{7.16}$$

$$\liminf_{n \to +\infty} \frac{\min\{\Delta_1 F(x, x_n), \Delta_1 F(x, x_n + 2^{-n})\}}{(n \log\log n)^{1/2}} \le -c. \tag{7.17}$$

By [10, Lemma 3.2] there are constants $0 < c' \le c'' < 1$ such that

$$c' \le \liminf_{n \to +\infty} \frac{|\{i \in \{1, \dots, n\} : X_i(x) = 0\}|}{n} \le \limsup_{n \to +\infty} \frac{|\{i \in \{1, \dots, n\} : X_i(x) = 0\}|}{n} \le c''$$

holds μ_0-a.s. x.

Let $\sigma(h) := \log_2(1/|h|)$ for $h \ne 0$. Using this and (6.1), there is $C < +\infty$ such that

$$\limsup_{n \to \infty} \frac{\sigma(x - x_n) + \sigma(x_n + 2^{-n} - x)}{\sigma(2^{-n})} \le C \quad \mu_0\text{-a.s.}x. \tag{7.18}$$

(7.16) and (7.18) imply the lower bound for (7.14). (7.17) and (7.18) imply the upper bound for (7.15).

---

[10] If

$$\limsup_{n \to \infty} Z^+_{k,n}(x) > 0 \text{ and } \limsup_{n \to \infty} Z^-_{k,n}(x) > 0 \quad \mu_0\text{-a.s.}x, \tag{**}$$

$$\limsup_{h \to 0} \frac{f_k(x+h) - f_k(x)}{|h|^c} = +\infty = -\liminf_{h \to 0} \frac{f_k(x+h) - f_k(x)}{|h|^c} \quad \mu_0\text{-a.s.}x.$$

If we can apply Lemma 7.8 to $\{Z_{k,n}\}_n$ for $k \ge 2$, (**) follows immediately and moreover Theorem 7.7 holds for $k \ge 2$.

Second we show the upper bound for (7.14) and the lower bound for (7.15). Assume $h > 0$. Applying Lemma 7.8 to $\{\sum_{i=1}^{n} X_i - E^{\mu_0}[X_i|\mathcal{F}_{i-1}]\}_n$ and $\{Y_i\}_i$,

$$l_x - l(x, x+h) + l_{x+h} - l(x, x+h) = O\left((l(x, x+h) \log\log l(x, x+h))^{1/2} \right) \text{ and}$$

$$Z_{1,l(x,x+h)}(x) = O\left((l(x, x+h) \log\log l(x, x+h))^{1/2} \right), h \to 0, h > 0, \ \mu_0\text{-a.s.}x.$$

Using (7.2)

$$\Delta_1 F(x, x+h) = O\left((\sigma(h) \log\log \sigma(h))^{1/2} \right), \ h \to 0, h > 0, \ \mu_0\text{-a.s.}x. \tag{7.19}$$

Now assume $h < 0$. Applying Lemma 7.8 to $\{\widetilde{Y}_i\}_i$,

$$\Delta_1 \widetilde{F}(y, y+h) = O\left((\sigma(h) \log\log \sigma(h))^{1/2} \right), \ h \to 0, h > 0, \ \widetilde{\mu}_0\text{-a.s.}y.$$

Let $T(y) := 1 - y$. By (2.4), $\widetilde{\mu}_0 = \mu_0 \circ T^{-1}$ and

$$\Delta_k \widetilde{F}(x, y) = \Delta_k F(1-x, 1-y), \quad x, y \in (0,1), \ k \geq 1. \tag{7.20}$$

Therefore

$$\Delta_1 F(x, x+h) = \Delta_1 \widetilde{F}(1-x, 1-x-h) = O\left((\sigma(h) \log\log \sigma(h))^{1/2} \right), \ h \to 0, h < 0, \ \mu_0\text{-a.s.}x. \tag{7.21}$$

(7.19) and (7.21) complete the proof of the upper bound for (7.14) and the lower bound for (7.15). □

Let

$$I_n(x) := \sum_{i=1}^{n} E\left[Y_i^2 | \mathcal{F}_{i-1}\right](x) \text{ and } \sigma(h, x) := I_{\lfloor \log_2(1/|h|) \rfloor}(x).$$

Corollary 7.9. *If μ_0 is absolutely continuous,*

$$\liminf_{h \to 0} \frac{\Delta_1 F(x, x+h)}{(\sigma(h,x) \log\log \sigma(h,x))^{1/2}} = -\sqrt{2} \qquad \mu_0\text{-a.s. } x. \tag{7.22}$$

Proof. Assume $x \notin D$ and $h > 0$. Using Lemma 7.2 and that

$$Z_{1,l_{x+h}}(x+h) \geq Z_{1,l(x,x+h)}(x+h) = Z_{1,l(x,x+h)}(x) + O(1),$$
$$\Delta_1 F(x, x+h) \geq \min\left\{ Z_{1,l_x}(x), Z_{1,l(x,x+h)}(x) \right\} + O(1).$$

Since $\displaystyle\lim_{h \to 0, h>0} \frac{l_x}{l(x, x+h)} = 1$,

$$\lim_{h \to 0, h>0} \frac{I_{l(x,x+h)}(x)}{\sigma(h,x)} = \lim_{h \to 0, h>0} \frac{I_{l_x}(x)}{\sigma(h,x)} = 1 \qquad \mu_0\text{-a.s.}x.$$

Therefore

$$\liminf_{h \to 0, h>0} \frac{\Delta_1 F(x, x+h)}{(\sigma(h,x) \log\log \sigma(h,x))^{1/2}} \geq -\sqrt{2} \qquad \mu_0\text{-a.s.}x.$$

By (7.20)

$$\liminf_{h \to 0, h<0} \frac{\Delta_1 F(x, x+h)}{(\sigma(h,x) \log\log \sigma(h,x))^{1/2}} \geq -\sqrt{2} \qquad \mu_0\text{-a.s.}x$$

in the same manner. Thus we have

$$\liminf_{h \to 0} \frac{\Delta_1 F(x, x+h)}{(\sigma(h,x) \log\log \sigma(h,x))^{1/2}} \geq -\sqrt{2} \qquad \mu_0\text{-a.s.}x.$$

If μ_0 is absolutely continuous,

$$\lim_{n \to \infty} \frac{\sigma(x - x_n, x)}{I_n(x)} = \lim_{n \to \infty} \frac{\sigma(x_n + 2^{-n} - x, x)}{I_n(x)} = 1 \qquad \mu_0\text{-a.s.}x.$$

Using this, Lemma 7.8 and (6.6), we have the upper bound of (7.22). □

If $b_1(t) = t + \dfrac{1}{2}$ and $c_0(t) = c_1(t) = 0$, by symmetry,

$$\partial_t F(0, x) = \partial_t F(0, 1-x), F(x) = 1 - F(1-x) \text{ and } \sigma(h, x) = \lfloor \log_2(1/|h|) \rfloor.$$

Hence

Corollary 7.10 (The original Takagi function case of [7, Theorem 5]). *If* $b_1(t) = t + \dfrac{1}{2}$ *and* $c_0(t) = c_1(t) = 0$*, the following hold* μ_0*-a.s.* x *:*

$$\limsup_{h \to 0} \frac{\Delta_1 F(x, x+h)}{(\log_2(1/|h|) \log\log\log_2(1/|h|))^{1/2}} = \sqrt{2} = -\liminf_{h \to 0} \frac{\Delta_1 F(x, x+h)}{(\log_2(1/|h|) \log\log\log_2(1/|h|))^{1/2}}.$$

Acknowledgement

The author would like to express his thanks to the referee for careful reading of the manuscript and many valuable suggestions. He also would like to express his thanks to E. de Amo for comments on an earlier version of this paper. This work was partly supported by Grant-in-Aid for JSPS fellows (24.8491).

References

[1] P. C. Allaart and K. Kawamura, Extreme values of some continuous, nowhere differentiable functions, *Math. Proc. Camb. Phil. Soc.* 140 (2006), no. 2, 269-295.
[2] P. C. Allaart and K. Kawamura, The improper infinite derivatives of Takagi's nowhere-differentiable function, *J. Math. Anal. Appl.* 372 (2010), no. 2, 656-665.
[3] E. de Amo, M. Díaz Carrillo and J. Fernández-Sánchez, Singular functions with applications to fractal dimensions and generalized Takagi functions, *Acta Appl. Math.* 119 (2012), 129-148.
[4] P. Billingsley, Van der Waerden's continuous nowhere differentiable function, *Amer. Math. Monthly* 89 (1982), 691.
[5] J. B. Brown, U. B. Darji and E. P. Larsen, Nowhere monotone functions and functions of nonmonotonic type, *Proc. Am. Math. Soc.* 127(1), (1999) 173-182.
[6] M. Hata and M. Yamaguti, The Takagi Function and Its Generalization, *Japan J. Appl. Math.* 1 (1984) 183-199.
[7] N. Kôno, On generalized Takagi functions, *Acta Math. Hung.* 49 (1987), 315-324.
[8] M. Krüppel, On the extrema and the improper derivatives of Takagi's continuous nowhere differentiable function, *Rostock. Math. Kolloq.* 62 (2007), 41-59.
[9] R. Mauldin and S. Williams, On the Hausdorff dimension of some graphs, *Trans. Amer. Math. Soc.* 298 (1986), 793-803.
[10] K. Okamura, Singularity results for functional equations driven by linear fractional transformations, *J. Theoret. Probab.* 27 (2014) 1316-1328.
[11] J. Paradís, P. Viader and L. Bibiloni, Riesz-Nágy singular fnuctions revisited, *J. Math. Anal. Appl.* 329 (2007) 592-602.
[12] G. de Rham, Sur quelques courbes définies par des équations fonctionalles, *Univ. e Politec. Torino. Rend. Sem. Mat.* 16 (1957), 101-113.
[13] W. F. Stout, A Martingale analogue of Kolmogorov's law of the iterated logarithm, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 15 (1970), 279-290.
[14] T. Takagi, A simple example of the continuous function without derivative, *Phys.-Math. Soc. Japan* 1 (1903), 176-177. *The Collected Papers of Teiji Takagi*, S. Kuroda, Ed., Iwanami (1973), 5-6.
[15] D. Williams, *Probability with martingales*, Cambridge University Press, 1991.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, JAPAN
E-mail address: `kazukio@kurims.kyoto-u.ac.jp`